Course 049064: Variational Methods in Image Processing

Notes 7: Segmentation II - Level Sets and Geodesic Active Contours

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1 Curve evolutions

1.1 Level Sets

This is a very popular numerical framework to evolve curves. We follow the explanations of [1], Section 4.3.3.

We would like to evolve a curve $C(t, q)$ in time using the following general equation:

$$\frac{\partial C}{\partial t} = FN, \ C(0, q) = C_0(q), \tag{1}$$

with N the unit normal to the curve (inward direction for closed curves), and F a general velocity term which can depend on the time, the curve and its derivatives: $F = F(t, c, c', c'')$.

The level set formulation [6] is based on the observation that *a curve can be seen as a zero-level of a function in a higher dimension.* There has been a vast literature on how to use level-sets to evolve curves and surfaces, see e.g. the books [4, 5, 7] and the references therein.

Suppose we have a function $u(t, x)$ such that for $x = C(t, q)$ we have the zero level-set for all t:

$$u(t, C(t, q)) = 0, \ \forall q, \forall t \geq 0. \tag{2}$$

Then if u is sufficiently smooth, we would like to differentiate the above equation with respect to t. Note that both t and $C(t, q)$ are functions of t. With some abuse of notations, let us denote $C(t) = (x(t), y(t))$ and write u as $u(t, x(t), y(t)$. Then we have

$$\frac{du}{dt} = u_t + u_x x_t + u_y y_t = u_t + \langle \nabla u, \frac{\partial C}{\partial t} \rangle.$$

Differentiating Eq. (2) with respect to t and using (1) we get:

$$\frac{\partial u}{\partial t} = -\langle \nabla u, FN \rangle. \tag{3}$$

We assume that u is negative inside the curve and positive outside, thus we have that the outward normal N is

$$N = -\frac{\nabla u}{|\nabla u|},$$

and therefore

$$\frac{\partial u}{\partial t}(t, C(t,q)) = |\nabla u| F. \tag{4}$$

We can extend F to be defined on the whole domain and have a standard evolution of function $u(x)$, $x \in \Omega$:

$$\frac{\partial u(t,x)}{\partial t} = |\nabla u| F. \tag{5}$$

As usual, Neumann boundary conditions are used: $\frac{\partial u}{\partial N} = 0$ on $\partial\Omega$.

For initialization, a signed distance function conveniently fulfils the requirements of Eq. (2) at time $t = 0$. We define:

$$\hat{d}(x,C) = \begin{cases} +d(x,C) & \text{if } x \text{ is outside } C, \\ -d(x,C) & \text{if } x \text{ is inside } C, \end{cases} \tag{6}$$

where $d(x,C)$ is the Euclidean distance to C. The initial condition is

$$u(0,x) = \bar{d}(x, C_0).$$

The main advantages of the level-sets method are:

1. The curve represented by $u(t,x)$ can change topology (break into several curves or merge). One does not need to take this topological changes into account numerically.

2. We can use a fixed spatial grid standard finite difference derivative approximations.

3. Geometric elements like normal vector and curvature can be easily expressed with respect to u.

4. The method can be easily extended to any dimension (for instance evolving surfaces in a volume).

Reinitialization: In practice, when we evolve a level set, we have at $t = 0$ the initial condition of a signed distance function from the initial curve. However, following the evolution the new function is not a signed distance function from the new curve anymore. In particular, its gradient can be unbounded and is the origin of serious numerical problems.

To overcome this difficulty, following several iterations of the evolution one "reinitialize" the evolved function, constructing a new distance function. The distance function can be computed in various ways, a fast popular method is to use an algorithm referred to as *fast marching method*, see more in [7].

2 Geodesic Active Contours

In [2] a very popular segmentation method was suggested by Caselles, Kimmel and Sapiro. It simplified the original active-contours model of Kass et al [3] and allowed the numerical advantages of the level-sets methods.

2.1 The model

It was observed that the elasticity term involving $C''(q)$ is not significant, as the first order term with $C'(q)$ already cause decrease in curvature. The active contour model is then simplified to:

$$E_{AC-simple}(C) = \int_0^1 |C'(q)|^2 dq + \lambda \int_0^1 g^2(|\nabla f(C(q))|)dq. \tag{7}$$

They suggested an alternative energy term which does not depend on the curve parametrization in the form of:

$$E_{GAC-1}(C) = \int_0^1 g(|\nabla f(C(q))|)|C'(q)|dq. \tag{8}$$

Having the relation $ds = |C'(q)|dq$, ds is the infinitesimal arc-length: $ds^2 = dx^2 + dy^2$, this can be written as

$$E_{GAC-2}(C) = \int_0^{L(C)} g(|\nabla f(C(q))|)ds, \tag{9}$$

where $L(C)$ is the Euclidean length of the curve:

$$L(C) = \int_0^1 |C'(q)|dq = \int_C ds.$$

Equation (9) can be interpreted as minimizing a new length definition, which takes into account the image to be segmented, and attains lower energy ("shorter curve") on regions with high gradient magnitude (edges).

The gradient descent in the normal direction is:

$$\frac{\partial C}{\partial t} = (\kappa g - \langle \nabla g, N \rangle) N. \tag{10}$$

For $g \equiv 1$ (no data from the image) we get the well-known mean curvature motion:

$$\frac{\partial C}{\partial t} = \kappa N. \tag{11}$$

This flow decreases the length as well as the total curvature, the number of zero crossings and the value of the maxima / minima curvature. An initial curve shrinks to a point in finite time with asymptotically circular shape.

An improvement to the flow of Eq. (10) is to add a constant term $\alpha > 0$ multiplying g to have:

$$\frac{\partial C}{\partial t} = (\kappa g - \langle \nabla g, N \rangle + \alpha g) N. \tag{12}$$

This increases the speed of convergence and allows better handling of non-convex objects (one should choose α large enough such that $\kappa + \alpha > 0$).

2.2 Level Set formulation

The level set expression of Eq. (12) is

$$\frac{\partial u}{\partial t} = \left((\kappa + \alpha) g + \langle \nabla g, \frac{\nabla u}{|\nabla u|} \rangle \right) |\nabla u|. \tag{13}$$

Using the expression for the curvature of Eq. (18) we can write it also as

$$\frac{\partial u}{\partial t} = \left(\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) + \alpha \right) g |\nabla u| + \langle \nabla g, \nabla u \rangle. \tag{14}$$

3 Short notions in differential geometry

Let $C(q) = (x(q), y(q))$ be a curve in $\mathbb{R}^2$.

The tangent vector at $C(q)$ is:

$$T(q) = C'(q) = (x'(q), y'(q)).$$

The normal vector at $C(q)$ is:

$$N(q) = (-y'(q), x'(q)).$$

The arc length is:

$$s(q) = \int_0^q \sqrt{(x'(r))^2 + (y'(r))^2} dr.$$

Curvature: The curvature, denoted by κ, is the magnitude of the rate of change of the unit tangent vector $T(s)$:

$$\kappa := \left| \frac{T(s)}{ds} \right|. \tag{15}$$

For circle of radius r it is the reciprocal of the radius: $\kappa = \frac{1}{r}$. For a general curve it could be understood as the reciprocal of the radius of the tangent circle.

For any parametrization we have:

$$\kappa(q) = \frac{x'(q)y''(q) - y'(q)x''(q)}{((x'(q))^2 + (y'(q))^2)^{3/2}}. \tag{16}$$

For level-sets, we assume the curve is on the iso-level line of a function $u(x, y)$, that is $u(x(s), y(s)) = const$. Then the curvature can be expressed as:

$$\kappa = \frac{(u_x)^2 u_{yy} + (u_y)^2 u_{xx} - 2u_x u_y u_{xy}}{((u_x)^2 + (u_y)^2)^{3/2}}. \tag{17}$$

which can be written also as:

$$\kappa = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{18}$$

References

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