Course 049064: Variational Methods in Image Processing

Notes 6: Segmentation I

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1 The goal of segmentation

The goal of segmentation is generally to generate a higher level representation / understanding of the image by partitioning it into homogeneous parts (or combining similar pixels to a set). The idea behind it is that objects are often combined of just a few segments. A good segmentation of the image often helps higher levels of computer vision algorithms.

The segmentation problem is very hard to solve. It is also not completely well defined, as homogeneous regions can be homogeneous not only in color but also in textures, patterns, orientations and other features. For specific applications, often some previous segmentation examples or direction from the user is needed to obtain meaningful results.

In this part we will learn some classical variational segmentation model such as Mumford-Shah [10], Chan-Vese [5] and Geodesic-Active-Contours [4].

2 Mumford-Shah

Mumford and Shah suggested in [10] the following model to solve the segmentation problem:

$$E_{MF}(u,L) = \int_{\Omega - L} (u-f)^2 dx + \alpha \int_{\Omega - L} |\nabla u|^2 dx + \beta \int_L ds, \qquad (1)$$

where, as usual, Ω is the image domain and f is the input image. Here we have a new concept $L \subset \Omega$ which is the set of discontinuities. The minimization is jointly over u and L. In other words one tries to find:

- A piecewise smooth function u which approximates f well.

- The function u is smooth (in the H^1 sense) everywhere but on the set of discontinuities L.
- The set L should be as short as possible.

This functional is very hard to minimize numerically. Ambrosio and Tortorelli therefore proposed in [1] an approximated functional with the following formulation:

$$E_{AT}(u,v) = \int_\Omega (u-f)^2 dx + \alpha \int_\Omega v^2 |\nabla u|^2 dx + \beta \int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2 \right) dx. \tag{2}$$

Here a new variable v was introduced, which has a similar role to the discontinuity set L. It has the following properties:

- It is close to 1 whenever u is sufficiently smooth.
- It is close to 0 near large gradients of u.
- It is smooth (in the H^1 sense).

As ε approaches 0 we get a more faithful approximation to the Mumford-Shah functional. More can be found in Th. 4.2.7 and 4.2.8 in [2] and in details in [1, 3]. A more sophisticated and efficient way to solve the M-S functional was suggested by Pock et al in [12].

2.1 Chan-Vese model

Chan and Vese in [5] suggested a binary model for object and background. The object and background are assumed to have more or less homogeneous color, or homogeneous average color for patterns and textures. Their concise formulation for the problem is to minimize the following energy:

$$E_{CV}(C, c_1, c_2) = \int_{\text{inside}(C)} (f - c_1)^2 dx + \int_{\text{outside}(C)} (f - c_2)^2 dx + \mu \int_C ds, \tag{3}$$

where the energy is minimized with respect to the closed curve C and the two unknown constants c_1, c_2.

A distinct advantage is the ability to go beyond edges and derivative and to segment also patterns and points with different concentrations (like the Europe lights map). In the original formulation [5] another energy term, the area inside the curve, was also suggested. In addition, different weights between the energy terms for object and background can be assigned.

The Chan-Vese functional can be viewed as a binary or piece-wise constant version of the Mumford-Shah functional. See more details in [2], p. 210.

Note that for a fixed curve C, optimizing for c_1, c_2 is done simply by computing the mean value of f inside and outside the curve, respectively. Let A_C be the set inside C, $|A_C| = \int_{A_C} dx$ is the area of A_C and $|\Omega| - |A_C|$ is area outside A_C. Then

$$\begin{aligned} c_1 = & \frac{1}{|A_C|} \int_{A_C} f(x)dx, \\ c_2 = & \frac{1}{|\Omega|-|A_C|} \int_{\Omega - A_C} f(x)dx, \end{aligned} \tag{4}$$

Algorithm: The general algorithm is:

1. Initialize with some curve $C = C_0$.
2. Compute c_1, c_2 according to Eq. (4).
3. For fixed c_1, c_2, minimize E_{CV} with respect to the curve C.
4. Repeat stages 2 and 3 until convergence.

A level set formulation was proposed to evolve the curve C and minimize the functional given in (3). More details on level sets will be given in the next lecture.

2.2 Convex model

Chan, Esedoglu and Nikolova suggested in [6] a convex model that finds a minimizer for the Chan-Vese functional, for a fixed c_1, c_2:

$$E_{CEN}(u, c_1, c_2) = \int_\Omega |\nabla u| dx + \lambda \int_\Omega \left((f(x) - c_1)^2 - (f(x) - c_2)^2 \right) u(x) dx, \tag{5}$$

where $0 \leq u(x) \leq 1$. The segmentation set A_C is recovered by thresholding u: $A_C := \{x \,|\, u(x) > 0.5\}$.

Algorithm: The algorithm is similar to the original Chan-Vese model, where one alternates between solving for u and recomputing the constants c_1, c_2. Solving u, however, is done not using level-sets, which is used to evolve curves, but using standard variational techniques for evolving functions.

Below we describe the gradient descent method suggested in the original paper [6]: One can introduce a convex penalty function $v(q)$ which is 0 for the interval $q \in [0, 1]$, descends linearly for $q < 0$ and ascends linearly for

$q > 1$, such that $v'(q) = -1$, $q < 0$, $v'(q) = 1$, $q > 1$ and 0 otherwise. Then the following gradient descent is used:

$$u_t = \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda s(x) - \alpha v'(u), \tag{6}$$

where

$$s(x) = (f(x) - c_1)^2 - (f(x) - c_2)^2.$$

2.3 Active Contours

The original active contours model of Kass-Witkin-Terzopoulos [8] inspired many segmentation algorithms. One can visualize their segmentation flow as a process where a band is closing-in around an object.

Let $C(q)$ be a closed curve defined by a parameter $q \in [0, 1]$:

$$C(q) = (C_1(q), C_2(q)), \;\; C'(q) = \left(\frac{dC_1}{dq}, \frac{dC_2}{dq}\right).$$

The assumption of closed curve means $C(0) = C(1)$. For s the gradient magnitude, we define an inverse edge detector (attains low values on edges) by:

$$g(s) = \frac{1}{1 + s^2}.$$

(this is similar to the P-M diffusion coefficient). Then the following active contour energy can be defined:

$$E_{AC}(C) = \int_0^1 |C'(q)|^2 dq + \beta \int_0^1 |C''(q)|^2 dq + \lambda \int_0^1 g^2(|\nabla f(C(q))|) dq. \tag{7}$$

The first two terms are called *internal energy* terms and the last term is an *external energy* term. The idea is that the "band" will be

- Short (first term penalizes length).
- Not too elastic (second term penalizes high curvature).
- On edges of the input image (third term penalizes smooth regions).

Some drawbacks of the method:

1. The functional depends on the curve parametrization (is not intrinsic).
2. The method assumes there is exactly one object. The curve cannot change topology.

3. One needs to solve a 4th order PDE, hard to implement numerically.

Using the foundations of the level-set formulation to evolve curves and surfaces in a stable manner a new level-set formulation was proposed by Caselles, Kimmel and Sapiro called Geodesic-Active-Contours [4]. This became an extremely popular method, and will be detailed in the next notes.

Some more reading on variational segmentation can be found e.g. in [7, 9, 11, 12].

References

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