Course 049064: Variational Methods in Image Processing

Notes 5: Spectral Total-Variation

Guy Gilboa

1 Spectral TV Decomposition

In [3, 4] a TV-spectral framework was formulated with a TV transform and inverse transform. It affords a different representation of the image and allows a continuous separation of scales with full edge and contrast preservation.

1.1 Preliminaries

1.1.1 Reminder: TV and TV-flow

The total variation for smooth u:

$$J_{\tilde{TV}} = \int_\Omega |\nabla u| dx. \tag{1}$$

For nonsmooth and possibly discontinuous u the distributional definition is used:

$$J_{TV} = \int_\Omega |Du| = \sup_z \left\{ \langle u, \mathrm{div}(z) \rangle, \ \|z\|_\infty \leq 1 \right\}. \tag{2}$$

The gradient descent which minimizes the total variation energy (with the input image f as initial condition), referred to also as *TV-flow*, is

$$u_t(t,x) = -\partial_u J_{TV}(u(t,x)) = \mathrm{div}\left(\frac{Du}{Du} \right), \ u(t=0,x) = f(x). \tag{3}$$

Existence and uniqueness of the TV-flow were proved in [1].

1.1.2 Convex one-homogeneous functionals

Convex one-homogeneous functionals have the following property for any $0 < \alpha \in \mathbb{R}$:

$$J(\alpha u) = \alpha J(u). \tag{4}$$

We denote a subgradient p which admits (**??**) $p(u) \in \partial J(u)$. For one-homogeneous functionals we have

$$J(u) = \langle u, p(u) \rangle, \tag{5}$$

and thus

$$J(v) \geq \langle p(u), v \rangle, \ \forall v. \tag{6}$$

We also have for all $\alpha > 0$

$$p(\alpha u) = p(u). \tag{7}$$

1.2 Nonlinear Eigenfunctions

Eigenfunctions of linear operators admit the following linear eigenvalue problem

$$Lu = \lambda u, \tag{8}$$

where L is a linear operator. Eigenfunctions of the graph-Laplacian have been used extensively to solve many signal processing, computer vision and machine-learning problems such as segmentation [7], clustering [6] and subspace clustering [5], dimensionality reduction [2], and more.

We view the subdifferential as a nonlinear operator and examine the following non-linear eigenvalue problem:

$$p(u) = \lambda u, \ p(u) \in \partial J(u). \tag{9}$$

Functions u admitting (9) are called eigenfunctions (of $J(u)$) with λ their corresponding eigenvalue.

When f is an eigenfunction, the TV-flow, Eq. (3), has a simple analytic solution:

$$u(t, x) = (1 - \lambda t)^{+} f(x), \tag{10}$$

where $(X)^{+} := \max(0, X)$.

1.3 TV Transform

The TV transform is defined by:

$$\phi(t; x) = u_{tt}(t; x) t, \tag{11}$$

where u_{tt} is the second time derivative of the solution $u(t;x)$ of the TV-flow (3). The inverse transform is:

$$f(x) = \int_0^\infty \phi(t;x)dt + \bar{f}, \tag{12}$$

where $\bar{f} = \frac{1}{\Omega}\int_\Omega f(x)dx$ is the mean value of the initial condition. Filtering is defined by a non-negative amplification function $H(t)$:

$$\phi_H(t;x) = \phi(t;x)H(t),$$

where the filtered image f_H is computed by reconstructing with ϕ_H (instead of ϕ) in Eq. (12). The spectrum $S(t)$ corresponds to the amplitude of each scale:

$$S(t) = \|\phi(t;x)\|_{L^1} = \int_\Omega |\phi(t;x)|dx. \tag{13}$$

Two significant results were shown in [4] for this transform:

- **Atoms as eigenfunctions:** Let $f(x)$ be an eigenfunction (admitting (9)) with eigenvalue λ, then the transform yields a single impulse, multiplied by $u(x)$, at $t = 1/\lambda$ and is zero for all other t:

$$\phi(t,x) = \delta(t - \frac{1}{\lambda})f(x). \tag{14}$$

- **Relations to TV-flow:** The TV flow solution $u(t_1)$ is a specific low-pass filter in this framework with the following scale-attenuation:

$$H_{TVF,t_1}(t) = \frac{t - t_1}{t},\ t_1 \leq t < \infty \text{ and zero otherwise.} \tag{15}$$

Ideal low-pass-filter (iLPF):

$$iLPF(t) := \int_t^\infty \phi(\tau)d\tau = u(t) + tp(u(t)). \tag{16}$$

Orthogonality of ϕ and u for any $t > 0$,

$$\langle \phi(t,x), u(t,x) \rangle = 0. \tag{17}$$

References

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