Course 049064: Variational Methods in Image Processing

Notes 3: Basic Functionals

Guy Gilboa

1 Reminder: some standard norms and spaces

1.1 Norms without derivatives

Very useful norms are the L^p norms, defined as

$$\|u\|_{L^p} \equiv \left(\int_\Omega (u(x))^p dx \right)^{1/p}.$$

Interesting special cases are:

1. L^2
$$\|u\|_{L^2} \equiv \sqrt{\int_\Omega (u(x))^2 dx}.$$

2. L^1
$$\|u\|_{L^1} \equiv \int_\Omega |u(x)| dx.$$

3. L^∞
$$\|u\|_{L^\infty} \equiv \max_{x \in \Omega} |u(x)|.$$

4. L^0
$$\|u\|_{L^0} \equiv \int_\Omega I(u(x)) dx,$$
where $I(q) = 1$ if $q \neq 0$ and 0 if $q = 0$.

The norms above are usually used as *fidelity terms*, penalizing distance to the input image.

Cauchy-Schwarz inequality: We define the L^2 inner product as

$$\langle u, v \rangle = \int_\Omega u(x)v(x)dx,$$

then we have the following inequality:

$$|\langle u, v \rangle| \leq \|u\|_{L^2}\|v\|_{L^2}.$$

Holder inequality: The more general case, for two norms L^p, L^q where $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$ we have:

$$|\langle u, v \rangle| \leq \|u\|_{L^p}\|v\|_{L^q}.$$

1.2 Semi-norms with derivatives

We assume a smooth signal (where the derivatives are well defined).

1. H^1
$$|u|_{H^1} \equiv \sqrt{\int_\Omega (\nabla u(x))^2 dx}.$$

2. TV
$$|u|_{TV} \equiv \int_\Omega |\nabla u(x)| dx.$$

These semi-norms are usually used as *smoothness terms*, penalizing non-smoothness of the signal.

In order to use them as norms the corresponding L^p norm is taken: H^1+L^2 or $TV + L^1$.

We will later see that weaker smoothness conditions are sufficient, so TV can be well defined also on discontinuous functions (using distributions to define weak derivatives).

1.3 Hilbert and Banach spaces

1. **Hilbert space** An inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

   (a) Inner product: $\langle u, v \rangle = \overline{\langle v, u \rangle}$, where $\overline{q}$ is the complex conjugate.

(b) The inner product is linear in its first argument and conjugate linear in the second argument (therefore linear for real functions). For $a, b \in \mathbb{R}$ we have $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$.

(c) The inner product induces a norm: $\|u\| = \sqrt{\langle u, u\rangle}$.

(d) The distance d between two functions u and v is defined by the norm: $d(u, v) = \|u - v\|$. The distance is symmetric and obeys the triangle inequality: $d(u, v) \leq d(u, w) + d(w, v)$.

2. **Banach space** is a complete normed vector space. It is more relaxed than the Hilbert space since it does not need an inner product to be defined in it.

1.4 Energy functionals

The two basic terms in energy minimization methods are:

$$E_{Total} = E_{Smoothness}(u) + \lambda E_{Fidelity}(u, f),$$

where f is the input image and u is the variable image for which the energy is minimized. The smoothness term involves derivatives and the fidelity terms is often an L^p norm.

Tikhonov regularization:

$$E_{Tik} = |u|^2_{H^1} + \lambda\|f - u\|^2_{L^2}.$$

Total variation regularization [4]:

$$E_{TV} = |u|_{TV} + \lambda\|f - u\|^2_{L^2}.$$

Good for images with uniform or white Gaussian noise.

Total variation deconvolution: A kernel H is assumed to convolve the input image: $f = g * H + n$, where g is the clean image, n is noise. The fidelity term changes to have the following energy:

$$E_{TV-deconv} = |u|_{TV} + \lambda\|f - u * H\|^2_{L^2}.$$

TV-L1 [3, 1]:

$$E_{TV-L1} = |u|_{TV} + \lambda\|f - u\|_{L^1}.$$

Good for removing outliers.

1.4.1 Co-area formula

We now show a fundamental relationship between length and TV.

The perimeter of a characteristics function of a shape A, χ_A is denoted by $\mathrm{Per}(\chi_A)$ (the length of the border ∂A).

Let us define a characteristic function $E_h(u) = 1$ if $u > h$ and 0 otherwise. The co-area formula is:

$$|u|_{TV} = \int_{-\infty}^{\infty} \mathrm{Per}(E_h(u))dh$$

a proof can be found for instance in Section 2.2.3 of [2].

References

[1] T.F. Chan and S. Esedoglu. Aspects of total variation regularized l 1 function approximation. *SIAM Journal on Applied Mathematics*, 65(5):1817–1837, 2005.

[2] T.F. Chan and J. Shen. *Image Processing and Analysis.* SIAM, 2005.

[3] M. Nikolova. A variational approach to remove outliers and impulse noise. *JMIV*, 20(1-2):99–120, 2004.

[4] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.