Course 049064: Variational Methods in Image Processing

Notes 1: Diffusion Processes - Part I

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1 Gaussian Scale-Space

The scale-space approach was suggested as a multi-resolution technique for image structure analysis.

1.1 Axiomatic approach

For low-level vision processing certain requirements were set in order to construct an uncommitted front-end [6]:

- Linearity (no previous model)
- Spatial shift invariance
- Isotropy
- Scale invariance (no preferred size)

1.2 Linear diffusion

The unique operator obeying all these requirements was a convolution with a Gaussian kernel. In order to be scale invariant, all scales were to be considered. Therefore the Gaussian convolution was to be applied to the input at all scales (standard deviation of Gaussian kernel ranging from 0 to ∞). The diffusion process (or heat equation) is equivalent to a smoothing process with a Gaussian kernel. In this context the linear diffusion equation was used:

$$u_t = c\Delta u, \quad u|_{t=0} = f, \quad c > 0 \in \mathbb{R}. \tag{1}$$

This introduced a natural continuous scale dimension t. For a constant diffusion coefficient $c = 1$, solving the diffusion equation (1) is analogous to convolving the input image f with a Gaussian of a standard deviation $\sigma = \sqrt{2t}$.

Important cues, such as edges and critical points, are gathered from information distributed over all scales in order to analyze the scene as a whole. One of the problems associated with this approach is that important structural features such as edges are smoothed and blurred along the flow, as the processed image evolves in time. As a consequence, the trajectories of zero crossings of the second derivative, which indicate the locations of edges, vary from scale to scale.

2 Spatially varying diffusion

For spatially-varying diffusion coefficient $c(x)$ the diffusion equation is:

$$u_t = \mathrm{div}(c(x)\nabla u), \tag{2}$$

2.1 Perona-Malik nonlinear diffusion

Perona and Malik (P-M) [4] addressed the issue of edge preservation by using the general divergence diffusion form to construct a nonlinear adaptive denoising process, where diffusion can take place with a spatially variable diffusion in order to reduce the smoothing effect near edges.

The general diffusion equation, controlled by the gradient magnitude, is of the form:

$$u_t = \mathrm{div}(c(|\nabla u|)\nabla u), \tag{3}$$

where in the P-M case, c is a positive decreasing function of the gradient magnitude. Two functions for the diffusion coefficient were proposed:

$$c_{PM1}(|\nabla u|) = \frac{1}{1 + (|\nabla u|/k_{PM})^2}$$

and

$$c_{PM2}(|\nabla u|) = \exp((|\nabla u|/k_{PM})^{-2}).$$

It turns out that both have similar basic properties (positive coefficient, nonconvex potentials, ability for some local enhancement of large gradients).

Results obtained with the P-M process paved the way for a variety of PDE-based methods that were applied to various problems in low-level vision (see [8] and references cited therein). Some drawbacks and limitations of the original model have been mentioned in the literature (e.g. [2], [13]). Catte et al.[2] have shown the ill-posedness of the diffusion equation, imposed by using the P-M diffusion coefficients, and proposed a regularized version wherein the coefficient is a function of a smoothed gradient:

$$u_t = \mathrm{div}(c(|\nabla u * g_\sigma|)\nabla u). \tag{4}$$

Note that although this formulation solved a deep theoretical problem associated with the Perona-Malik process, the characteristics of this process essentially remained. Weickert et al. [12] investigated the stability of the P-M equation by spatial discretization, and proposed [5] a generalized regularization formula in the continuous domain.

Two other useful diffusion coefficients that fall within the Perona-Malik framework, Eq. (3), are the Charbonnier diffusivity [3]:

$$c_{Ch}(|\nabla u|) = \frac{1}{\sqrt{1 + (|\nabla u|/k)^2}}$$

and TV-flow [1], which minimizes the total-variation energy of Rudin-Osher-Fatemi [7] without the fidelity term:

$$c_{TV}(|\nabla u|) = \frac{1}{|\nabla u|}.$$

The TV-flow coefficient approaches infinity as $|\nabla u| \to 0$ and the process has some interesting unique properties, as we will see later. It is often approximated by a Charbonnier-type formula: $c_{TV\epsilon}(|\nabla u|) = \frac{1}{\sqrt{\epsilon^2 + |\nabla u|^2}}$.

2.2 Weickert's anisotropic diffusion

Weickert suggested a further generalization, preferring smoothing along the local dominant direction (for each pixel), where a tensor diffusion coefficient is used.

2.2.1 Tensor Diffusivity

For oriented flow-like structures, such as fingerprints, truly anisotropic processes are called for. Processes emerging from Eq. (3) are controlled by a scalar diffusion coefficient $c(x, y, t)$. This permits a spatially varying process that can also change throughout the evolution, but is basically isotropic, that is, locally the process acts the same in all directions (in the regularized version, see [12]). Weickert [11, 10] suggested an effective anisotropic scheme using a tensor diffusivity. The diffusion tensor is derived by manipulation of the eigenvalues of the smoothed structure tensor $\mathcal{J}_\sigma = g_\sigma * (\nabla u_\sigma \nabla u_\sigma^T)$.

This technique results in strong smoothing along edges and low smoothing across them. In relatively homogenous regions without coherent edges, the process approaches linear diffusion with low diffusivity. The semilocal nature of the process may extract information from a neighborhood of radius proportional to σ. This enables completion of interrupted lines and enhances flow-like structures.

2.3 Characteristics of the flow

The anisotropic diffusion of Weickert still possesses some important characteristics which ensures stability and well behavior of the evolution (see more in Weickert's book [9]):

- Mean value is preserved
- Existence and uniqueness
- Extremum principle
- Well posedness, continuous dependence on the initial image.
- Gray-level shift invariance
- Translation invariance
- Rotation invariance

We will give some more details on these properties in the next part.

3 Math definitions and main equations

Notations and Symbols

Ω	Image domain
$\partial\Omega$	Image boundary
$f(x)$ or f	Input image
$u(x)$ or u	Solution, evolved image
$n(x)$ or n	Noise
g_σ	Gaussian of standard deviation σ
$\mathbf{n}$	Direction normal to the boundary
$\mathcal{J}(u)$	Structure tensor of u
$x = (x_1, .., x_N)$	Spatial coordinate
t	Time (scale)
dt	Time step
∇	Gradient
Δ	Laplacian
div	Divergence
X_y or $\partial_y X$ or $\frac{\partial X}{\partial y}$	Partial derivative of X with respect to y
X_{yy} or $\partial_{yy} X$	Partial second derivative of X with respect to y
$\vec{Y}$	Vector Y of functions $(Y_1(x), .. Y_N(x))$
$\vec{Y}^T$	Transpose of vector $\vec{Y}$
$*$	Convolution

3.1 Basic differential operators

All definitions are given for the N-dimensional case, $u \in \mathbb{R}^N$. In images $N = 2$, in volumes (e.g. medical imaging) $N = 3$.

Partial derivative

$$u_{x_i} := \lim_{h \to 0} \frac{u(x_1, .., x_i + h, .., x_N) - u(x_1, .., x_i, .., x_N)}{h}$$

Gradient

$$\nabla u := (u_{x_1}, .., u_{x_N}), \; N\text{-dimensional vector.}$$

Divergence

$$\operatorname{div} \vec{v} := \sum_{i=1}^{N} \partial_{x_i} v_i(x)$$

Laplacian

$$\Delta u := \text{div}(\nabla u) = \sum_{i=1}^{N} u_{x_i x_i}$$

Structure Tensor

$$\mathcal{J}(u) := (\nabla u)(\nabla u)^T = |_{N=2} \begin{pmatrix} u_x^2 & u_x u_y \\ u_x u_y & u_y^2 \end{pmatrix}$$

Smoothed structure-tensor: $\mathcal{J}_\sigma(u) := g_\sigma * \mathcal{J}(u)$.

Boundary Conditions (BC)
Dirichlet: $u(x) = F(x)$, $\forall x \in \partial\Omega$.
Neumann: $\frac{\partial u(x)}{\partial \mathbf{n}} = F(x)$, $\forall x \in \partial\Omega$, where $\mathbf{n}$ is the direction normal to the boundary ($\frac{\partial u(x)}{\partial \mathbf{n}} = \nabla u(x) \cdot \mathbf{n}(x)$).
In image processing one usually uses zero Neumann BC: $\frac{\partial u(x)}{\partial \mathbf{n}} = 0$.

3.2 Common Scale-Spaces

The scale space is evolved by the general diffusion equation $u_t = \text{div}(c(x)\nabla u)$, where the diffusion coefficient $c(x)$ controls the characteristics of the evolution. Some examples (from most stable and smoothing process to least):

Name	**c(x)**	**Comment**
Linear	1	equivalent to Gaussian convolution
Charbonnier et al.	$\frac{1}{\sqrt{1+\lvert\nabla u\rvert^2/k^2}}$	Minimizes a strictly convex functional
TV-flow	$\frac{1}{\lvert\nabla u\rvert}$	Needs special numerical schemes
Weickert	Tensor D	Coherence enhancing, depends on smoothed structure tensor.
Perona-Malik	$\frac{1}{1+\lvert\nabla u\rvert^2/k^2}$ or $e^{-\lvert\nabla u\rvert^2/k^2}$	Maintains strong edges
FAB (GSZ)	$\frac{1}{\sqrt{1+(\lvert\nabla u\rvert/k_f)^2}} - \frac{\alpha}{1+(\lvert\nabla u\rvert/k_b)^2}$	Stable sharpening flow
Linear inverse diffusion	-1	Unstable sharpening flow

References

[1] F. Andreu, C. Ballester, V. Caselles, and J. M. Mazn. Minimizing total variation flow. *Differential and Integral Equations*, 14(3):321–360, 2001.

[2] Francine Catté, Pierre-Louis Lions, Jean-Michel Morel, and Tomeu Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM Journal on Numerical analysis*, 29(1):182–193, 1992.

[3] P. Charbonnier, L. Blanc-Feraud, G. Aubert, and M. Barlaud. Two deterministic half-quadratic regularization algorithms for computed imaging. In *Proc. IEEE ICIP '94*, volume 2, pages 168–172, 1994.

[4] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *PAMI*, 12(7):629–639, 1990.

[5] E. Radmoser, O. Scherzer, and J. Weickert. Scale-space properties of nonstationary iterative regularization methods. *Journal of Visual Communication and Image Representation*, 8:96–114, 2000.

[6] Bart M Ter Haar Romeny. Introduction to scale-space theory: Multiscale geometric image analysis. In *Verlag. First International Conference on Scale-Space theory*. Citeseer, 1996.

[7] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[8] B M ter Haar Romeny(Ed.). *Geometry Driven Diffusion in Computer Vision.* Kluwer Academic Publishers, 1994.

[9] J. Weickert. *Anisotropic Diffusion in Image Processing.* Teubner-Verlag, Stuttgart, Germany, 1998.

[10] J. Weickert. Coherence-enhancing diffusion of colour images. *IVC*, 17:201–212, 1999.

[11] Joachim Weickert. Coherence-enhancing diffusion filtering. *International Journal of Computer Vision*, 31(2-3):111–127, 1999.

[12] Joachim Weickert and Brahim Benhamouda. A semidiscrete nonlinear scale-space theory and its relation to the peronamalik paradox. In *Advances in Computer Vision*, pages 1–10. Springer, 1997.

[13] Ross T Whitaker and Stephen M Pizer. A multi-scale approach to nonuniform diffusion. *CVGIP: Image Understanding*, 57(1):99–110, 1993.