Course 049064: Variational Methods in Image Processing

Exercise 4: Nonlocal operators, convex analysis and spectral TV

Exercise due on 13.2.2017

Please send a PDF file containing a full solution (analytic and experimental results) and the Matlab files folder to Guy Gilboa, guy.gilboa@ee.technion.ac.il. The subject of the mail should be **"Exercise 4"**. In the mail please write the full names + ID of the participants

1 Analytic exercises (50 points)

1. *Convex analysis:*

   (a) Prove that the $TV - \varepsilon$ energy $J(u) = \int_\Omega \sqrt{|\nabla u(x)|^2 + \varepsilon^2} dx$ is convex for any $\varepsilon > 0$.

   (b) Legender-Fenchel transform: find the Legender-Fenchel transform (Notes 10, Eq. (3)) $f^*(m)$ of $f(x) = x^2$. Show that $f^{**} = f$. Direction: find the supremum by taking the derivative with respect to m and equate to 0.

   (c) Convex one homogeneous functionals: prove that for J a convex one homogeneous functional we get $p(\alpha u) = p(u)$ for any $\alpha > 0$ (Notes 10, Eq. (11)). Direction: use Eqs. (9) and (10) in Notes 10 and choose special cases of v.

2. *Spectral TV:*

   (a) Construct an ideal TV high-pass-filter $HPF(t)$ which is a function of $u(t)$, t, $p(u(t))$ and f (but not of $\phi(t)$). That is, for $f(x)$ an eigenfunction with eigenvalue λ (Eq. (13)), with mean value

$\bar{f} = 0$, we get $u(t;x) = f(x)$ for $t > 1/\lambda$ and 0 otherwise. Prove your claim.

(b) Let $\hat{S}^2(t) = \langle f, \phi(t) \rangle$. Show a Parseval-type rule: $\int_0^\infty \hat{S}^2(t)dt = \|f\|_{L^2}^2$ for f with mean value 0.

3. *Nonlocal calculus:*

   (a) Show the divergence theorem, Eq. (10) in Notes 9.

   (b) **Creative:** For nonsymmetric weights $w(x,y) \neq w(y,x)$, suggest a nonlocal gradient and divergence which are adjoint, that is obey Eq. (9), Notes 9. Try to find a true non-symmetric formulation, do not symmetrize the weights. What is the nonlocal Laplacian in this case?

2 Matlab experiments (50 points)

All references of equations and pages are to Notes 9.

1. *Nonlocal diffusion*: Implement nonlocal diffusion, Eq (20):

$$u_t = \Delta_w u, \ \ u|_{t=0} = f.$$

   Use explicit method. Compute the weights according to the description on p. 8, use the suggested patch size, search window and m. Use $a = 2$, $h = \sigma$. Add white Gaussian noise of standard deviation $\sigma = 10$. Use the two images given on the website for Ex. 4 and one image of your own. Stop the evolution near the best PSNR. Compare the best PSNR of this process to the Perona-Malik process you have implemented in the first exercise. Note, the PSNR for gray scale images between [0..255] is

$$PSNR := 10 \log_{10} \left(255^2 / \|g - u\|_{L^2}^2 \right),$$

   where g is the clean image (without noise) and u is the filtered result.