Course 049064: Variational Methods in Image Processing

Exercise 1: Linear and nonlinear diffusion

Exercise due on 5.12.2016

Please send a PDF file containing a full solution (analytic and experimental results) and the Matlab files folder to Guy Gilboa, guy.gilboa@ee.technion.ac.il. The subject of the mail should be **"Exercise 1"**. In the mail please write the full names + ID of the participants (it is recommended to do the work in pairs).

1 Analytic exercises (30 points)

1. **Linear diffusion (15 points)**:
   (a) For $u \in \mathbb{R}$ (1D, unbounded domain)
   - Show that a Gaussian convolution with the initial condition solves the linear diffusion equation: $u_t = u_{xx}$, $u(t = 0; x) = f(x)$. That is $u(t; x) = f(x) * g_{\sigma(t)}(x)$, where $*$ denotes convolution.
   - Find the function F relating the evolution time t to the Gaussian standard deviation σ: $\sigma = F(t)$.
   - Viewing the process as a linear filter, plot the frequency response of this filter at times $t = 1, 4, 25$ (on the same graph).

   (b) Let $f \in \mathbb{R}$, $f(x) = \sin(\omega x)$. Let $u(t = 0; x) = f(x)$, $u_t = u_{xx}$, $t \in [0, \infty)$. For $t = \Delta t$:

   (i) Write the explicit backward-difference in time expression of $u(\Delta t; x)$. Keep the spatial coordinates continuous.

(ii) Write the analytic solution of the approximation of $u(\Delta t; x)$ given in(i).

(iii) Does the solution admit the extremum principle in all cases? if not, can additional conditions be added such that the extremum principle holds? Prove this.

2. **Perona-Malik (15 points)**:

   (a) It was shown in class that the original P-M diffusion coefficient $c(s) = 1/(1 + (s/k)^2)$ can have both forward and inverse diffusion parts, depending on s/k. Can you find a single bounded expression $c_{new}(s)$, with $0 \leq c_{new} \leq 1$, which is an edge preserving coefficient (less diffusion near edges) however it is guaranteed not to have an inverse diffusion part in all cases? Prove your claim.

   (b) Prove two properties of the P-M equation in the 2D case:

      i. Invariance to a constant shift in the gray levels (so for initial conditions $f(x)$ and $f(x) + const$ the solutions are $u(x)$ and $u(x) + const$, respectively).

      ii. Solutions at all times have the mean value of the initial condition:
$$\frac{1}{|\Omega|} \int_{\Omega} u(t; x) dx = \frac{1}{|\Omega|} \int_{\Omega} f(x) dx.$$

2 Matlab experiments (70 points)

For the tasks below use the Sun-girl image for linear diffusion and the Parrot image for the Perona-Malik part. If you want to check more variations you can use an image of your own in addition (optional).

1. Linear diffusion **(30 points)**: implement linear diffusion and analyze the input and result image (use as input Sun-girl image):

   (a) Compute and plot graphs of the following measures as a function of time:

      i. Standard deviation.

      ii. 3 histograms of the gradient magnitude $|\nabla u|$ in the image at times $t = 0.1, 1, 10$. Show all histograms on the same scale.

   Explain the trends and why they occur.

   (b) Compare diffusion to Gaussian convolution with equivalent sigma. If there is a difference, explain why.

2. Perona-Malik **(40 points)**: Write the function implementing nonlinear Perona-Malik diffusion with Charbonnier diffusivity (PMC). Use as input Parrot image. Use the following diffusion coefficient $c(s) = 1/\sqrt{1 + (s/k)^2}$, where $s = |\nabla u|$. Note the square-root, this is different than what was shown in class.

   (a) **1D denoising (15 points)**. A step with noise. Add noise, std=0.1 (where the step is between [0,1]). Compare denoising with linear diffusion and with PMC. Choose appropriate threshold k for PMC.

   (b) **2D denoising (25 points)**:

      i. Add noise of std=10 and std=20, denoise with several k values. How does k relate to the noise std?

      ii. Perform the experiments with two different time steps: $\Delta t = 0.2$ and $\Delta t = 1$. Explain the results.

      iii. Take noise with std=10. Denoise with $k = 10$. Compute the SNR curve, $SNR(t)$ as a function of the evolution time t, what do you see?

      iv. *Creative part (10 points).* Suggest a method / criterion to stop automatically the evolution at a good time-point (SNR-wise or according to another quality measure). Check your idea.