Course 049064: Variational Methods in Image Processing

Exercise 1: Linear and nonlinear diffusion

Exercise due on 27.11.2017

Please send a PDF file containing a full solution (analytic and experimental results) and the Matlab files folder to Tal Feld, stmfeld@gmail.com. The subject of the mail should be **"Exercise 1"**. In the mail please write the full names + ID of the participants (it is highly recommended to do the work in pairs).

1 Analytic exercises (30 points)

1. **Linear diffusion (15 points)**:

   (a) For $u \in \mathbb{R}$ (1D, unbounded domain), show that a Gaussian convolution with the initial condition solves the linear diffusion equation: $u_t = u_{xx}$, $u(t = 0; x) = f(x)$. That is $u(t; x) = f(x) * g_{\sigma(t)}(x)$, where $*$ denotes convolution.

   (b) Prove that linear diffusion is invariant to translation and rotation. Assume a signal in $\mathbb{R}^2$.

   (c) Let $f \in \mathbb{R}$, $f(x) = \sin(\omega x)$. Let $u(t = 0; x) = f(x)$, $u_t = u_{xx}$, $t \in [0, \infty)$. For $t = \Delta t$:

      (i) Write the explicit backward-difference in time expression of $u(\Delta t; x)$. Keep the spatial coordinates continuous.

      (ii) Write the analytic solution of the approximation of $u(\Delta t; x)$ given in(i).

(iii) Does the solution admit the extremum principle in all cases? If not, can additional conditions be added such that the extremum principle hold? Prove this.

2. **Perona-Malik (15 points)**:

   (a) It was shown in class that the original P-M diffusion coefficient $c(s) = 1/(1 + (s/k)^2)$ can have both forward and inverse diffusion parts, depending on s/k. Can you find an edge preserving coefficient (less diffusion near edges) which is guaranteed not to have an inverse diffusion part for any signal? Prove your claim.

   (b) Prove two properties of the P-M equation in the 2D case:

      i. Invariance to a constant shift in the gray levels (so for initial conditions $f(x)$ and $f(x) + const$ the solutions are $u(x)$ and $u(x) + const$, respectively).

      ii. Solutions at all times have the mean value of the initial condition:
$$\frac{1}{|\Omega|}\int_{\Omega} u(t;x)dx = \frac{1}{|\Omega|}\int_{\Omega} f(x)dx.$$

2 Matlab experiments (70 points)

For the tasks below use the Sun-girl image for linear diffusion and the Parrot image for the Perona-Malik part. If you want to check more variations you can use an image of your own in addition (optional).

1. Linear diffusion **(30 points)**: implement linear diffusion and analyze the input and result image (use as input Sun-girl image):

   (a) Compute and plot three graphs of the following measures as a function of time:

      i. Mean value of the image.

      ii. Standard deviation.

      iii. Total variation.

      Explain the trends and why they occur.

   (b) Compare diffusion to Gaussian convolution with equivalent sigma. If there is a difference, explain why.

   (c) Inverse diffusion. Blur an image without noise and with noise std=5. Perform linear inverse diffusion for a short time, compare the results.

2. Perona-Malik **(40 points)**: Write the function implementing nonlinear Perona-Malik diffusion (use as input Parrot image). Use the diffusion coefficient $c(s) = 1/(1 + (s/k)^2)$, where $s = |\nabla u|$.

   (a) **1D denoising (15 points)**. A a step with noise. Add noise, std=0.1 (where the step is between [0,1]). Compare denoising with linear diffusion and with P-M. Choose appropriate threshold k for P-M.

   (b) **2D denoising (25 points)**:

      i. Add noise of std=10 and std=20, denoise with several k values. How does k relate to the noise std?

      ii. Perform the experiments with two different time steps: $\Delta t = 0.2$ and $\Delta t = 1$. Explain the results.

      iii. Take noise with std=10. Denoise with $k = 10$. Compute the SNR curve, $SNR(t)$ as a function of the evolution time t, what do you see?

      iv. *Creative part (10 points).* Suggest a method / criterion to stop automatically the evolution at a good time-point (SNR-wise or according to another quality measure). Check your idea.