$$\bar{f}(q) = 2\pi \int_0^\infty \tilde{f}(r) J_0(2\pi qr) qrdr$$
$$\tilde{f}(r) = 2\pi \int_0^\infty \bar{f}(q) J_0(2\pi qr) qrødq$$

**Characteristics:**

Scalar: \( f(ax) \) \(\leftrightarrow\) \( \frac{1}{a^2} F(q/a) \)

Linearity: \( f(r) + g(r) \) \(\leftrightarrow\) \( F(q) + G(q) \)

Shift: This contradicts symmetry and is forbidden

Power Eqn: \( \int_0^\infty |f(r)|^2 r dr = \int_0^\infty |F(q)|^2 q dq \)

Convolution: \( \int_0^\infty \int_0^\infty \tilde{f}(r) \tilde{g}(R-r) r dr'd\sigma \leftrightarrow F(q)G(q) \)

- \( R^2 = r^2 + r'^2 - 2rr'\cos\sigma \)

**Examples:**

1) \( 1/r \leftrightarrow 1/q \)
2) \( e^{-x^2} \leftrightarrow e^{-q^2} \)
5. STRUCTURE AND OPERATION OF THE EYE

The lens is flexible allowing for a variable focal length. The lens is circular in shape and is made up of water and proteins. The pupil's diameter varies between \( d = 2 - 10 \text{mm} \), thus controlling the amount of light entering the eye. The focal length of the lens is controlled by muscles and varies between 18 - 24mm. The retina is made up of two types of light sensitive receptors which operate for either day or night vision. There exists a large concentration of daylight receptors on the fovea. The line connecting the fovea and the centre of the pupil is called the visual axis and differs from the optical axis. The blind spot has no receptors and we can not see any of the light falling on it.

5.1. Geometrical Optics of a Thin Lens

The lens is a body with a refractive index which differs from the surroundings and has two radii. The characteristic equation is:

\[
\frac{1}{f} = (n-1) \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

where \( f = \) focal length
\( n = \) refractive index

We characterize the lens using \( f \). It can have a negative value. We define:

The basic optical equation is given by:

\[
\frac{1}{f} = \frac{1}{u} + \frac{1}{v}
\]

where \( u = \) position of the object relative to the lens. (+) implies that the object lies in front of the lens while -ve implies that the object lies behind the lens.
\( v = \) position of the image relative to the lens. (+) implies that the image lies behind the lens, and visa versa.
Examples:

I) A positive lens with \( f = 20 \text{cm} \).

a) \( u \rightarrow \infty \), plane wave - the rays arrive in parallel.
\[
\frac{1}{f} = \frac{1}{v} \Rightarrow v = 20 \text{cm}.
\]

b) Source distance \( f \) in front of the lens \( (u = 0) \).
\[
\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = 0, \Rightarrow v = \infty.
\]

c) Object at \( u = 2f = 40 \text{cm} \).
\[
\frac{1}{v} = \frac{1}{f} - \frac{1}{2f} = \frac{1}{2f} \Rightarrow v = 2f.
\]

II) A negative lens which disperses the light.

a) \( u = \infty \)
\[
\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{1}{f} \Rightarrow v = f = -20
\]

b) \( u = -f \)
\[
\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{1}{20} - \frac{1}{20} = \frac{1}{10}
\]

5.2. The Lens in the Human Eye

The lens of the eye has a variable \( f \) and has object distance limits of :
\[
\text{Umin} \leq U < \infty \quad \text{where Umin is age dependent.}
\]

<table>
<thead>
<tr>
<th>AGS</th>
<th>Umin (cm)</th>
<th>For a far image ( u \rightarrow \infty ), ( f = v = 24 \text{mm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7</td>
<td>For a far object:</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>For a near object:</td>
</tr>
<tr>
<td>30</td>
<td>14</td>
<td>\text{Assume Umin} = 1.2 \text{cm} \text{ V = 2.4cm} \Rightarrow f = 20 \text{mm}</td>
</tr>
<tr>
<td>40</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

Thus we can see that \( f \) varies between 20 and 24mm. We may find a shorter focal length in a strong eye.

In human vision we find two main types of problem:-
Hyperopic Vision

The lens is not convex enough.
An adult does not see close.
Here we add a positive lens.

Myopic Vision

Here the lens is too convex.
Youth does not see far.
Here we add a negative lens.

5.2.1. Correction of Hyperopic and Myopic Vision

1) Myopic Correction

This is the short sightedness and is corrected by adding a divergent lens.
We take the furthermost point that can be clearly seen to be at Xmax.
If U = ∞ and V = -Xmax.
Then: \(1/f = 1/-X_{max} + 1/∞ = 1/X_{max}\)
Thus \(f = X_{max}\)

In Dioptries: \(D = 1/f\) (in meters).
\(X_{max} = 100\text{cm} \Rightarrow D = -1\)
\(X_{max} = 50\text{cm} \Rightarrow D = -2\)

2) Hyperopic Correction

This is corrected by adding a convergent lens. If the nearest point that can be clearly seen is at Xmin, then if U = 25cm and V = -Xmin, we get:
\(1/f = 1/25 + 1/-X_{min} \Rightarrow f = \frac{25}{X_{min}/(X_{min} - 25)}\)

In Dioptries: \(D = 1/f\) (in meters).
\(X_{min} = 100\text{cm} \Rightarrow f = 32.5\text{cm} \Rightarrow D = +3\)
\(X_{min} = 50\text{cm} \Rightarrow f = 50.0\text{cm} \Rightarrow D = +2\)
\(X_{min} = 25\text{cm} \Rightarrow f = ∞ \Rightarrow D = 0\)
5.3. The Pupil

The pupil has a diameter varying between 2 to 10mm and is influenced by the amount of light hitting the eye. The illumination ratio is 1:25. Experimentally it was found that:

\[ d = 5 - 3 \tanh(0.4 \log_{10} B) \]

where \( \tanh = (e^x - e^{-x})/(e^x + e^{-x}) \)

Eg. \( B = 10^4 \text{ [cd/m}^2\text{]} \Rightarrow d = 7.8\text{mm} \)

\( B = 10^2 \text{ [cd/m}^2\text{]} \Rightarrow d = 3.0\text{mm} \)

5.4. Illumination of the Retina

The illumination of the retina is influenced by \( B \) and the diameter of the pupil.

Creation of an Image by the Lens:

For 3-D characteristics:

\[ \frac{dy}{dy'} = \frac{u}{v} \text{ and } \frac{dz}{dz'} = \frac{u}{v} \]

Thus:

\[ \frac{dy}{dy'} \frac{dz}{dz'} = \frac{dA}{dA'} = \left(\frac{u}{v}\right)^2 \]

If \( v \) is fixed then:

\[ dA / dA' \left(\frac{u}{v}\right) = 1/v^2 = \text{a constant.} \]

In the lens of the eye \( f \) varies while \( v \) remains constant. The eye has a refractive index of \( n = 4/3 \), thus,

\[ \frac{dy}{dy'} = n \frac{u}{v} \]

Thus for the eye:

\[ dA / dA' \left(\frac{u}{v}\right)^2 = \left(\frac{n}{v}\right)^2 = \text{a constant.} \]
For example:-

\[ v = 2.2 \text{cm} \text{ and } n = 4/3 \Rightarrow (n/v)^2 = 0.36 \text{cm}^4 \]

From this we can calculate what part of the field of vision is covered by 1mm in the retina.

\[ \frac{dy}{dy'} = \frac{n \nu}{u} \Rightarrow dy/u = \frac{(n \nu)}{u} \text{ dy'} = \tan \sigma = \sigma \]

where \( \sigma \) is the angle which is subtended by \( dy \), and is small (radians).

\[ \sigma = (n \nu) \text{ dy'} \text{ (max) } = 0.06 \text{ rad } = 3.4^\circ. \]

The density of receptors at the centre of the retina is approximately 150000 per mm\(^2\) \(\Rightarrow\) = 387 receptors per mm.

Thus we find 387 receptors per 3.4\(^\circ\).

1\(^\circ\) in the field of vision falls on to 387/3.4\(^\circ\) receptors, thus we find:-

\[ \approx \text{120 receptors per degree in the field of vision.} \]

Thus the Nyquist sampling frequency permits us to sample at rates of up to 60 cycles per degree [CPD]. The size of each receptor is \( \approx 1/120 \) of a degree.

5.4.1. Resolution Experiments (Acuity)

a) We bring the two lines together and see where they appear to become a single line, is fall on the same pixel in the retina.

b) Here we bring the lines closer until they appear to become a single line.
We find that the results in this case are 10 times better than those obtained in the first case. They both fall on the same pixel in the retina (hyperacuity).

5.5. Illumination of Retina by a Lambertian plane with Luminance \( B \).

For a Lambertian Plane:

\[ B = \frac{dlc}{dAcos \sigma} \Rightarrow dlc = B \ dAcos \sigma = B \ dA \ (cos \sigma = 1, \ \sigma = 0) \]

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Illumination of the pupil plane:
\[ dE = d\sigma \cos\alpha x^2 = B \, dA/x^2 \]

Flux passing through the pupil:
\[ dF = dE \cdot s = B \, s \, dA/x^2 \text{where } s = \text{area of the pupil.} \]

Losses due to the interior of the eye:
\[ dF' = dF \cdot \tau \quad \text{where } \tau_{\text{int}} = 0.5 \]

This flux falls on area \( dA' \), thus:
\[ E' = \frac{dF'}{dA'} = \tau \cdot \frac{dF}{dA} \]
\[ = \tau \cdot \frac{(dA/x^2) \, dA'}{dA} \quad B = \tau \cdot \alpha \cdot B \quad \text{where } \alpha = dA/x^2 \cdot dA' \]
\[ E' = \tau \cdot \alpha \cdot B \quad \alpha = (n/v)^2 = 0.36 \text{cm}^2 \]
\[ \gamma \] constant dependent on the eye.

Illumination of the retina is directly dependent on \( B \cdot s \) and has units in troland.

\[ B \, \text{[candle/m}^2\text{]} \cdot s \, \text{[mm}^2\text{]} = B \, s \, \text{[troland]} \]

5.6. Maxwellian View

Illumination of the lens:

\[ E = I/(kx)^2 \]

The flux falling on the plane \( dA = dx \, dy \) is:
\[ dF = E \cdot dA = I \cdot dA/(kx)^2 \]

This implies that all flux enters the eye. Assuming the losses in the lens are negligible the total amount of flux hitting the retina is given by:
\[ dF' = \tau dF = \tau I \cdot dA/(kx)^2 \]
\[ E = \frac{dF'}{dA'} = \tau \cdot \left( \frac{I}{k^2} \right) \cdot (dA/x^2 \cdot dA') \]
\[ = \tau \cdot \alpha \cdot I/k^2 \]
There is equivalence between \(1/a^2\) and \(B\) as from the previous case.

5.7. Stiles Crawford Phenomenon

We can shift the source, \(I\), such that the light spot falls on different points on the retina. It was found that if the light was first projected straight into the centre of the pupil and then near the edge of the pupil with diameter 8mm the brightness perception went down 5 times.

For every observer we represent the optimal point with \(n = 1\), and depending on the distance from this point we find a decrease in sensitivity according to:

\[
\log_{10} n = -ar^2
\]

where \(r = \text{distance from pupil in mm}\)
\(a = \text{coefficient dependant on the colour of the light.}\)
\(a = 0.05 \text{ for white light - photopic vision}\)
\(a = 0 \text{ for scotopic vision.}\)

\[
\eta = e^{ar^2} \quad r \text{ [mm]}
\]

An alternate expression is given by:

\[
\eta = 1 - 0.085r^2 + 0.002r^4 \quad \text{i.e. series expansion.}
\]

Thus the area of the pupil is not directly related to the illumination level. Instead of the geometric area,

\[
s = \pi \left(\frac{d}{2}\right)^2
\]

we can calculate the effective area.

\[
Se = 2 \int_0^{r_e} \eta \ r \ dr \quad (s \text{ for } \eta = 1)
\]

Integrating:

\[
Se = \left(2 \pi/2.3 \ a \right) \left(1 - e^{ar^2}\right)
\]

\[
Se = \pi \ \frac{d^2}{4} (1 - 0.085d/\eta + 0.0024d^2/\eta)
\]

B Se has units of [Effective Troland] and allows us to determine brightness sensitivity more accurately.

Given \(B\), what is the illumination of the retina?

* From \(B\) we calculate the diameter of the pupil.
\[
d = 5 - 3\tanh(0.4\log_{10}B)
\]

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* From \( d \) we calculate \( S_e \) from one of the above equations.
* We then calculate \( B \cdot S_e \).

Examples:

<table>
<thead>
<tr>
<th>( B \ [\text{cd/m}^2] )</th>
<th>( d \ [\text{mm}] )</th>
<th>( S_e \ [\text{mm}^2] )</th>
<th>( B \cdot S_e \ [\text{cd} \cdot \text{mm}^2] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^4 )</td>
<td>7.8</td>
<td>47</td>
<td>( 7 \times 10^4 ) **</td>
</tr>
<tr>
<td>0.1</td>
<td>6.1</td>
<td>30</td>
<td>20.</td>
</tr>
<tr>
<td>10</td>
<td>3.9</td>
<td>12</td>
<td>10 ( B \cdot S_e )</td>
</tr>
<tr>
<td>100</td>
<td>2.0</td>
<td>7</td>
<td>6.5 ( B \cdot S_e )</td>
</tr>
</tbody>
</table>

* Limit of photopic vision
** Scotopic vision (no effect)

5.8. Differences Between the Real and Standard (CIE) Observer

The CIE curves differ from the real values obtained for the average observer.

5.8.1. For Photometry:

* A distribution exists in the \( K(\lambda) \) curve.
* The maximum from CIE curve is 555nm, while in practice it varies between 550 - 570nm ± 4nm.
* The following parameters influence the results:
  - The size of the field and the position of the illuminated area.
  - Background illumination influences brightness perception in the field.
  - Adaptation to strong or weak light.
  - Seasonal changes of up to 5% on the \( K(\lambda) \) curve.
  - Purkinje effect affects both photopic and scotopic vision and mesopic vision (\( 10^2 \) - \( 1 \) \( \text{cd/m}^2 \)).
  - The age of the observer. Blue sensitivity decreases with age.
* Superposition leads to inconsistencies of up to 30%.

5.8.2. For Colorimetry:

* The CIE curves differ from the measured curves.
* We find colour blindness in 8% of men and 0.5% of women.
* Types of colour blindness:
Dichromatism:

I) Protanomalous - 1% low sensitivity to red.
   Deuteranomalous - 5% low sensitivity to green.

II) Protanopes - 1% blind to red.
   Deuteranopes - 1% blind to green.

Monochromatism:

III) Monochromats - 10^4%

**Type I**

Identification: Ask the subject to match 535nm (green) and 670nm (red) to 589nm.

The log of the relationship between the two colours = \( \log(p(670)/p(535)) \)

**Type II**

This creates area of equal colour (Confusion Loci).

**Type III**

Here there are no colour curves and we only find hues of grey.

These characteristics can be formed into a hierarchical table where each level can better or match the performance of all the layers below it but can not match the performance of the layers above.

A normal person can differentiate colours which cause both types I and II problems. Type I can handle colours which cause problems for II (Protanopes and Deuteranopes respectively).